

Time Domain Simulations of EMRIs using Finite Element Methods

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Abstract. This is a brief report on time-domain numerical simulations of extreme-mass-ratio binaries based on finite element methods. We discuss a new technique for solving the perturbative equations describing a point-like object orbiting a non-rotating massive black hole and the prospects of using it for the evaluation of the gravitational self-force responsible of the inspiral of these binary systems. We also discuss the perspectives of transferring this *technology* to the more astrophysically relevant case of a central rotating massive black hole.

INTRODUCTION

Extreme-mass-ratio binaries in the inspiral stage of their evolution (EMRIs) are considered to be a primary source of gravitational radiation to be detected by LISA. They consist of a “small” object (SO), such a main sequence star, a stellar mass black hole, or a neutron star orbiting a massive black hole (MBH), and the mass ratios of interest lie in the range $10^{-3} - 10^{-6}$. Their study is expected to provide crucial information regarding MBHs at the center of galaxies, tests of the validity of general relativity, possibility of discerning among different theories of galaxy formation, etc.

Since the signal from EMRIs will be buried in the LISA *noise*, it is crucial to have a good theoretical understanding of their evolution in order to produce accurate waveform templates to be used in data analysis schemes for detection and extraction of physical information. There are several efforts in this direction, all based on perturbative methods given the small mass ratios involved. They differ mainly in the way in which the *backreaction* on the SO is handled. In order of increasing complexity they are: (i) *Klugde* waveforms. The concept is to use approximations that allow quick generation of waveform templates [1]. (ii) The *adiabatic* approximation. It is based on the idea that the long-term evolution may be approximated by the *dissipative* part of the gravitational *self-force* [2]. (iii) The self-force approach. It consists in evaluating the full self-force on the SO [3], and from there to compute the modified motion and the associated waveforms. Approaches (i) and (ii) may provide useful templates for EMRI detection but it is unlikely that they can be used to extract relevant physical information (in particular the first multipole moments of the MBH). Moreover, it has been suggested [4] that the adiabatic approximation may not work as well as it was thought, in the sense that the conservative part of the self-force may have a significant effect in the long-term evolution. Therefore, there is a good motivation to pursue the approach (iii).

The study of the dynamics of EMRIs via the self-force approach involves a number of challenges [5] and can be divided into the following three stages: (A) The computation of the gravitational perturbations produced by the SO in the *background* spacetime of the MBH, in particular at the SO location. (B) Solving the equations of motion for the SO including its own gravitational field [given by solving (A)]. (C) Extraction of meaningful physical information, in particular the gravitational waveforms. None of these parts has been completely solved. The type of difficulties that appear depends strongly on whether the SO is modeled as a point-like object or as an extended object. Here we adopt the first possibility, which is the one more commonly used. Our discussion will focus on solving (A) with the perspective of using the result for (B). This means to carry out (A) in a gauge suitable for the self-force calculation. For instance, in the Regge-Wheeler gauge, the reconstruction of metric perturbations at the particle location involves singular terms. On the contrary, the Lorentz gauge is a very convenient one since a scheme for regularizing the self-force, the *mode-sum* scheme [6], has been formulated on it.

COMPUTING THE PERTURBATIONS PRODUCED BY A POINT-LIKE OBJECT USING FINITE ELEMENT METHODS

Point (A) consists in solving for the perturbations produced by a point-like object orbiting a Kerr black hole (MBHs at the galactic centers are expected to have high spins). However, due to the complexity of the rotating case, it is advisable to deal first with the case of a non-rotating Schwarzschild black hole. We discuss later the case of Kerr. In any case, in order to solve the corresponding perturbative equations we need to resort either to additional perturbative methods or to numerical techniques. While the first ones may have difficulties with orbits in the strong field regime, the second ones should be capable of providing the solution for any type of orbit. Among the possible numerical methods, time-domain schemes seem better suited than frequency-domain ones. In particular for highly eccentric orbits, the orbits with more astrophysical relevance, for which the frequency-domain approach has more difficulties since one has to sum over a large number of modes to obtain a good accuracy.

For a non-rotating MBH, the linear equations governing the perturbations can be decomposed in spherical harmonics. Each harmonic decouples from the rest, and so do polar and axial modes. As a result each mode obeys inhomogeneous equations that involve only dependence on time and a *radial* coordinate x , i.e. they are one-dimensional partial differential equations (PDEs). Of crucial importance is the structure of the sources of these inhomogeneous equations. They are generated by the energy-momentum distribution of the particle and can be represented as follows

$$\mathcal{S}^{\ell m} = F_0^{\ell m} \delta [x - x_p(t)] + F_1^{\ell m} \delta' [x - x_p(t)] + F_2^{\ell m} \delta'' [x - x_p(t)] , \quad (1)$$

where $x_p(t)$ is the radial motion of the particle, $(F_0^{\ell m}, F_1^{\ell m}, F_2^{\ell m})$ are functions of t and x , and δ , δ' , and δ'' denote the Dirac delta distribution and its first and second derivatives respectively. When the perturbations are described by pure metric perturbations, like in the Lorentz gauge, we have $F_1^{\ell m} = F_2^{\ell m} = 0$. When they are described by variables of the type of the Regge-Wheeler and Zerilli master functions only $F_2^{\ell m} = 0$. Finally, if

the perturbations are described by curvature-type variables we have all the terms. This representation of the source terms shows their singular structure. In the best case, the solution of the perturbative equations will be continuous but with discontinuous first radial derivatives. In the worst case ($F_2^{\ell m} \neq 0$), the solution will diverge as we approach the particle location.

When solving numerically the perturbative equations it is then crucial that we discretize appropriately the source terms. This is a non-trivial task but there are some prescriptions that produce reliable results. As a matter of fact, these prescriptions are based on integral forms of the equations, which allow the use of the known properties of the Dirac delta distribution, avoiding the introduction of *artificial* regularizations of it like a Gaussian packet. The first such prescription was proposed in [7] and is based on a finite differences scheme that resembles the procedures used to derive finite volume algorithms. Its main drawback is that it cannot be easily generalized to higher-dimensional domains, and hence to the Kerr case.

We have recently proposed [8] a different prescription based on FE methods for the spatial discretization. In brief (see details in [8, 9]), since the spatial domain is one-dimensional (the radial direction), the domain discretization consists in a division into disjoint intervals, our *elements*. Then, we assign to each element (interval) a finite-dimensional functional space in order to approximate locally the solution of our equations. Typically, these functional spaces are made out of polynomials, and the accuracy and convergence properties of the resulting FE algorithm depend strongly on how we choose them. We can approximate the solution of our PDEs, say ψ , by an expansion in nodal functions, n_i [$n_i(x_j) = \delta_{ij}$], constructed from the functional spaces:

$$\psi_h(t, x) = \sum_{i=0}^N \psi_i(t) n_i(x). \quad (2)$$

Then, one transforms the PDEs into their *weak* form, an integral form that consists in multiplying the equations by an arbitrary test function, integrating over the spatial domain, and applying integration by parts to eliminate second spatial derivatives while incorporating non-essential boundary conditions (for instance, Sommerfeld boundary conditions). We denote the weak form of the equations by $\mathcal{E}[\phi, \psi] = 0$, where ϕ is the test function. In a Galerkin-type FE formulation, the discretized equations are obtained by imposing the vanishing of all the *residuals*:

$$\mathcal{E}_i \equiv \mathcal{E}[n_i, \psi_h] = 0 \quad (i = 0, \dots, N). \quad (3)$$

Introducing (2) into (3) we get as many equations as independent functions $\psi_i(t)$ we have in (2). These equations are ordinary differential equations that can be solved by using adequate integrators. The important point for our discussion is that the equations that we obtain from (3) will contain terms of the type

$$\int dx n_i(x) \mathcal{S}^{\ell m}, \quad (4)$$

which can be easily evaluated by using the properties of the Dirac delta distribution. And this is essentially how the FE method provides a prescription for the singular source

terms in (1). This prescription depends on our choice of nodal functions, and hence on the domain discretization. There are two important factors: (i) The degree of the polynomials of the functional spaces of the FE discretization. (ii) The location of the particle in the mesh. Essentially whether it is located at a node or inside an element.

The degree of the polynomials determines the degree of convergence of the FE method. For instance, for linear elements ($n_i \sim ax + b$) and smooth functions we expected second-order convergence. However, the singular structure of our source terms may deteriorate it. Actually, linear elements will not be able to resolve the δ'' distribution since the action of it on any linear element will give zero. As a consequence, if we want to evaluate accurately self-forces from the perturbations, we need to choose carefully the type of elements.

The location of the particle plays a role in the following sense: If the particle is located inside an element, then the integrals of the source terms are evaluated using the properties of the Dirac delta distributions. This is simple in the sense that we only need to know the particular element that contains the particle. However, the convergence rate may be lower than expected, which happens in the case where $F_1^{\ell m} \neq 0$. On the contrary, if we locate the particle in a node, we can assign two values of the perturbative variables (and their derivatives depending on the exact singular structure of the source terms) to this node and impose there the *jumps* in the derivatives of these variables or the variables themselves. These jumps can be analytically derived from the perturbative equations [8]. The advantage of this method is that it preserves the expected convergence rate of the FE method. The drawbacks are that implicit evolution algorithms may be subject to a Courant-Friedrichs-Lax stability condition, and that in order to keep the particle always at a node we need to implement moving mesh techniques.

We have performed simulations using the techniques described here, using also refinement of the meshes, in the Regge-Wheeler gauge [8] (in this case $F_2^{\ell m} = 0$). We found an excellent agreement with other previous calculations of the energy and angular momentum luminosities for all kinds of orbits, including highly eccentric ones. Regarding the different numerical implementations discussed above, we found that their performance depends strongly on the type of orbit that we are considering, in the sense that the *best* scheme changes among the different types of orbits. We also found that the mesh refinement and moving mesh techniques help substantially in reducing the computational resources used by the simulations without damaging the accuracy. On the other hand, we have also performed simulations in the Lorentz gauge (in this case $F_1^{\ell m} = F_2^{\ell m} = 0$), which is a suitable one for self-force computations. So far, the only known results in this gauge are only for circular orbits and can be found in [10]. We have been able to perform simulations for all kinds of orbits and the results will be published elsewhere.

EXTENSION TO THE CASE OF A ROTATING MBH

One of the advantages of the FE method is that many of the techniques discussed above for the case of Schwarzschild can be transferred to the case of Kerr, in contrast with previous approaches. The PDEs governing the perturbations of Kerr can be two- or three-dimensional, depending on whether we factor out the dependence on the azimuthal an-

gle. In any case, prescriptions for the regularization of the corresponding source terms can be derived in exactly the same way as in the Schwarzschild case if the particle is always inside an element. In contrast, in order to use the techniques corresponding to the case in which the particle is at a node we would need to derive a new framework. An important point to be taken into account is the fact that the solutions of the perturbative equations, in the case in which we separate the azimuthal dependence, diverge logarithmically as we approach the particle. This fact will require either the subtraction *a priori* of the singular part of the solution or a careful numerical regularization.

Another important feature of the FE method that may be of crucial importance for the rotating case is the natural way in which refinement schemes can be implemented. Taking into account that we will be dealing either with two- or three-dimensional domains, refinement may be a necessity in order to carry out simulations at the required accuracy. Here, it is important to mention that, to a certain extent, we can import to the Kerr case the refinement schemes used in the Schwarzschild case. The way of doing this is to use for the Kerr case quadrilateral/hexahedral elements, since for these elements some classes of nodal functions can just be obtained from the one-dimensional ones by means of *tensorial* products.

In summary, the main challenges for the Kerr case seem to be the subtraction of the logarithmic singularities and the implementation of an efficient mesh refinement scheme. The rest of ingredients in the simulations do not seem to present additional difficulties with respect to the non-rotating case, in which we have already acquired a considerable amount of experience.

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